一般相対性理論 演習問題 第 4 回

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I.

ここでは $\dot{a} = \mathrm{d}a/\mathrm{d}t$ のように表記する。

変分原理より

$$\delta \int \frac{1}{2}\left[-\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + a^2\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + b^2\left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2 + c^2\left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2\right]\mathrm{d}\lambda = 0$$

を得る。変分をばらして

$$\int\left[-\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\delta t + a^2\frac{\mathrm{d}x}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\delta x + a\,\delta a\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + b^2\frac{\mathrm{d}y}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\delta y + b\,\delta b\left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2 + c^2\frac{\mathrm{d}z}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\delta z + c\,\delta c\left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2\right]\mathrm{d}\lambda = 0 \tag{1.1}$$

となる。

ここで、$\delta t_1 = \delta t_2 = 0$ を用いて

$$\int_1^2\left[-\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\delta t\right]\mathrm{d}\lambda = -\left[\frac{\mathrm{d}t}{\mathrm{d}\lambda}\,\delta t\right]_1^2 + \int_1^2\left[\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)\delta t\right]\mathrm{d}\lambda = \int_1^2\left[\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2}\,\delta t\right]\mathrm{d}\lambda$$

となる。同様に $\delta x_1 = \delta x_2 = 0$ を用いて

$$\int_1^2\left[a^2\frac{\mathrm{d}x}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\delta x\right]\mathrm{d}\lambda = \left[a^2\frac{\mathrm{d}x}{\mathrm{d}\lambda}\,\delta x\right]_1^2 - \int_1^2\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(a^2\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)\delta x\right]\mathrm{d}\lambda = \int_1^2\left[-a^2\frac{\mathrm{d}^2 x}{\mathrm{d}\lambda^2}\,\delta x - 2a\dot{a}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}x}{\mathrm{d}\lambda}\,\delta x\right]\mathrm{d}\lambda$$

となる。y, z についても同様である。

以上を式 (1.1) に代入して

$$\int\left[\begin{aligned}\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2}\,\delta t - a^2\frac{\mathrm{d}^2 x}{\mathrm{d}\lambda^2}\,\delta x - 2a\dot{a}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}x}{\mathrm{d}\lambda}\,\delta x + a\dot{a}\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2\delta t \\ - b^2\frac{\mathrm{d}^2 y}{\mathrm{d}\lambda^2}\,\delta y - 2b\dot{b}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}y}{\mathrm{d}\lambda}\,\delta y + b\dot{b}\left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2\delta t \\ - c^2\frac{\mathrm{d}^2 z}{\mathrm{d}\lambda^2}\,\delta z - 2c\dot{c}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}z}{\mathrm{d}\lambda}\,\delta z + c\dot{c}\left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2\delta t\end{aligned}\right]\mathrm{d}\lambda = 0$$

を得る。この式は、任意の $\delta t, \delta x, \delta y, \delta z$ について成り立つので、それぞれの係数は 0 でなければならない。それぞれの係数を整理して

$$\begin{aligned}&\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + a\dot{a}\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + b\dot{b}\left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2 + c\dot{c}\left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2 = 0 \\ &-a^2\frac{\mathrm{d}^2 x}{\mathrm{d}\lambda^2} - 2a\dot{a}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}x}{\mathrm{d}\lambda} = 0, \qquad -b^2\frac{\mathrm{d}^2 y}{\mathrm{d}\lambda^2} - 2b\dot{b}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}y}{\mathrm{d}\lambda} = 0, \qquad -c^2\frac{\mathrm{d}^2 z}{\mathrm{d}\lambda^2} - 2c\dot{c}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}z}{\mathrm{d}\lambda} = 0\end{aligned}$$

が成り立つ。

これが測地線方程式 $\dfrac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\lambda^2} + \Gamma^\sigma_{\mu\nu}\dfrac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\dfrac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0$ と一致するから、接続係数 $\Gamma^\sigma_{\mu\nu}$ の 0 でない成分は

$$\begin{aligned}&\Gamma^t_{xx} = a\dot{a}, \qquad \Gamma^t_{yy} = b\dot{b}, \qquad \Gamma^t_{zz} = c\dot{c} \\ &\Gamma^x_{tx} = \Gamma^x_{xt} = \frac{\dot{a}}{a}, \quad \Gamma^y_{ty} = \Gamma^y_{yt} = \frac{\dot{b}}{b}, \quad \Gamma^z_{tz} = \Gamma^z_{zt} = \frac{\dot{c}}{c}\end{aligned}$$

■注意　$\mu \neq \nu$ のとき、$\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$ であるから、測地線方程式において $\Gamma^\sigma_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}, \Gamma^\sigma_{\nu\mu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ の項が両方現れることを考慮して、$\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$ は $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}$ の係数の半分にしなけらばならない。

II.

ここでは $A_t \coloneqq \partial A/\partial t$ のように表記する。
変分原理より

$$\delta\int\frac{1}{2}\left[-A\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2+B\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2+r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2+r^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\right]\mathrm{d}\lambda=0$$

を得る。変分をばらして

$$\int\left[\begin{array}{c}-A\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta t-\frac{1}{2}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\delta A\\ +B\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta r+\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2\delta B\\ +r^2\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta\theta+r\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2\delta r\\ +r^2\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta\phi+r\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta r+r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta\theta\end{array}\right]\mathrm{d}\lambda=0$$

全問と同様に部分積分を実行して

$$\int\left[\begin{array}{c}A\frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2}\delta t+A_t\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\delta t+A_r\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\delta t-\frac{1}{2}A_t\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\delta t-\frac{1}{2}A_r\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\delta r\\ -B\frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2}\delta r-B_t\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\delta r-B_r\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\delta r+\frac{1}{2}B_t\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2\delta t+\frac{1}{2}B_r\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2\delta r\\ -r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2}\delta\theta-2r\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\delta\theta+r\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2\delta r\\ -r^2\sin^2\theta\frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2}\delta\phi-2r\sin^2\theta\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\delta\phi-2r^2\sin\theta\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\delta\phi+r\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta r+r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta\theta\end{array}\right]\mathrm{d}\lambda=0$$

この式は、任意の $\delta t,\delta r,\delta\theta,\delta\phi$ について成り立つので、

$$\begin{aligned}&A\frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2}+\frac{1}{2}A_t\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2+A_r\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda}+\frac{1}{2}B_t\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2=0\\ &-B\frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2}-\frac{1}{2}A_r\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2-B_t\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda}-\frac{1}{2}B_r\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2+r\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2+r\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2=0\\ &-r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2}-2r\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}+r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\\ &-r^2\sin^2\theta\frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2}-2r\sin^2\theta\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}-2r^2\sin\theta\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=0\end{aligned}$$

これが測地線方程式 $\dfrac{\mathrm{d}^2x^\sigma}{\mathrm{d}\lambda^2}+\Gamma^\sigma_{\mu\nu}\dfrac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\dfrac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}=0$ と一致するから、接続係数 $\Gamma^\sigma_{\mu\nu}$ の 0 でない成分は

$$\begin{aligned}&\Gamma^t_{tt}=\frac{A_t}{2A},\quad \Gamma^t_{tr}=\Gamma^t_{rt}=\frac{A_r}{2A},\quad \Gamma^t_{rr}=\frac{B_t}{2A}\\ &\Gamma^r_{tt}=\frac{A_r}{2B},\quad \Gamma^r_{tr}=\Gamma^r_{rt}=\frac{B_t}{2B},\quad \Gamma^r_{rr}=\frac{B_r}{2B},\quad \Gamma^r_{\theta\theta}=-\frac{r}{B},\quad \Gamma^r_{\phi\phi}=-\frac{r\sin^2\theta}{B}\\ &\Gamma^\theta_{r\theta}=\Gamma^\theta_{\theta r}=\frac{1}{r},\quad \Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta,\\ &\Gamma^\phi_{r\phi}=\Gamma^\phi_{\phi r}=\frac{1}{r},\quad \Gamma^\phi_{\theta\phi}=\Gamma^\phi_{\phi\theta}=\cot\theta\end{aligned}$$

III.

ここでは $\dot{a} = \mathrm{d}a/\mathrm{d}t$ と表記する。

■接続係数　変分原理より

$$\delta\int\frac{1}{2}\left[-\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2+\frac{a^2}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2+a^2r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2+a^2r^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\right]\mathrm{d}\lambda=0$$

を得る。変分をばらして

$$\int\left[\begin{array}{c}
-\dfrac{\mathrm{d}t}{\mathrm{d}\lambda}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\delta t\\
\underline{+\dfrac{a^2}{1-kr^2}\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\delta r}+\dfrac{a\dot{a}}{1-kr^2}\left(\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2\delta t+\dfrac{ka^2r}{(1-kr^2)^2}\left(\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2\delta r\\
\underline{+a^2r^2\dfrac{\mathrm{d}\theta}{\mathrm{d}\lambda}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\delta\theta}+a\dot{a}r^2\left(\dfrac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2\delta t+a^2r\left(\dfrac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2\delta r\\
\underline{+a^2r^2\sin^2\theta\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\delta\phi}+a\dot{a}r^2\sin^2\theta\left(\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta t+a^2r\sin^2\theta\left(\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta r+a^2r^2\sin\theta\cos\theta\left(\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta\theta
\end{array}\right]\mathrm{d}\lambda=0$$

全問と同様に部分積分を実行して

$$\int\left[\begin{array}{c}
\dfrac{\mathrm{d}^2t}{\mathrm{d}\lambda^2}\delta t\\
\underline{-\dfrac{a^2}{1-kr^2}\dfrac{\mathrm{d}^2r}{\mathrm{d}\lambda^2}\delta r-\dfrac{2a\dot{a}}{1-kr^2}\dfrac{\mathrm{d}t}{\mathrm{d}\lambda}\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\delta r-2\dfrac{ka^2r}{(1-kr^2)}{}^2\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\delta r}\\
+\dfrac{a\dot{a}}{1-kr^2}\left(\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2\delta t+\dfrac{ka^2r}{(1-kr^2)^2}\left(\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2\delta r\\
\underline{-a^2r^2\dfrac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2}\delta\theta-2a\dot{a}r^2\dfrac{\mathrm{d}t}{\mathrm{d}\lambda}\dfrac{\mathrm{d}\theta}{\mathrm{d}\lambda}\delta\theta-2a^2r\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\dfrac{\mathrm{d}\theta}{\mathrm{d}\lambda}\delta\theta}\\
+a\dot{a}r^2\left(\dfrac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2\delta t+a^2r\left(\dfrac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2\delta r\\
\underline{-a^2r^2\sin^2\theta\dfrac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2}\delta\phi-2a\dot{a}r^2\sin^2\theta\dfrac{\mathrm{d}t}{\mathrm{d}\lambda}\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\delta\phi-2a^2r\sin^2\theta\dfrac{\mathrm{d}r}{\mathrm{d}\lambda}\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\delta\phi-2a^2r^2\sin\theta\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}\lambda}\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\delta\phi}\\
+a\dot{a}r^2\sin^2\theta\left(\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta t+a^2r\sin^2\theta\left(\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta r+a^2r^2\sin\theta\cos\theta\left(\dfrac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\delta\theta
\end{array}\right]\mathrm{d}\lambda=0$$

この式は、任意の $\delta t, \delta r, \delta\theta, \delta\phi$ について成り立つので、

$$\frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2}+\frac{a\dot{a}}{1-kr^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2+a\dot{a}r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2+a\dot{a}r^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2=0$$

$$-\frac{a^2}{1-kr^2}\frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2}-\frac{2a\dot{a}}{1-kr^2}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda}-\frac{ka^2r}{(1-kr^2)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2+a^2r\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2+a^2r\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2=0$$

$$-a^2r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2}-2a\dot{a}r^2\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}-2a^2r\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}+a^2r^2\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2=0$$

$$-a^2r^2\sin^2\theta\frac{\mathrm{d}^2\phi}{\mathrm{d}\lambda^2}-2a\dot{a}r^2\sin^2\theta\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}-2a^2r\sin^2\theta\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}-2a^2r^2\sin\theta\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=0$$

これが測地線方程式 $\dfrac{\mathrm{d}^2x^\sigma}{\mathrm{d}\lambda^2}+\Gamma^\sigma_{\mu\nu}\dfrac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\dfrac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}=0$ と一致するから、接続係数 $\Gamma^\sigma_{\mu\nu}$ の 0 でない成分は

$$\Gamma^t_{rr}=\frac{a\dot{a}}{1-kr^2},\qquad \Gamma^t_{\theta\theta}=a\dot{a}r^2,\qquad \Gamma^t_{\phi\phi}=a\dot{a}r^2\sin^2\theta$$

$$\Gamma^r_{tr}=\Gamma^r_{rt}=\frac{\dot{a}}{a},\qquad \Gamma^r_{rr}=\frac{kr}{1-kr^2},\qquad \Gamma^r_{\theta\theta}=-(1-kr^2)r,\quad \Gamma^r_{\phi\phi}=-(1-kr^2)r\sin^2\theta$$

$$\Gamma^\theta_{t\theta}=\Gamma^\theta_{\theta t}=\frac{\dot{a}}{a},\qquad \Gamma^\theta_{r\theta}=\Gamma^\theta_{\theta r}=\frac{1}{r},\qquad \Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta$$

$$\Gamma^\phi_{t\phi}=\Gamma^\phi_{\phi t}=\frac{\dot{a}}{a},\qquad \Gamma^\phi_{r\phi}=\Gamma^\phi_{\phi r}=\frac{1}{r},\qquad \Gamma^\phi_{\theta\phi}=\Gamma^\phi_{\phi\theta}=\cot\theta$$

■$k=0$ の場合　まだ

■$k=-1$ の場合　まだ

■$k=1$ の場合　まだ

IV.

問 II の結果に

$$
\begin{aligned}
&A = 1 - \frac{2GM}{r}, \qquad A_t = 0, \quad A_r = \frac{2GM}{r^2} \\
&B = \left(1 - \frac{2GM}{r}\right)^{-1}, \quad B_t = 0, \quad B_r = -\frac{2GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-2}
\end{aligned}
$$

を代入して

$$
\begin{aligned}
&\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + \frac{2GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda} = 0 \\
&\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} + \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 - r\left(1 - \frac{2GM}{r}\right)\left\{\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\right\} = 0 \\
&\frac{\mathrm{d}^2 \theta}{\mathrm{d}\lambda^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 = 0 \\
&\frac{\mathrm{d}^2 \phi}{\mathrm{d}\lambda^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} + 2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = 0
\end{aligned}
$$